Function

A *function of one independent variable* is a relation in the form $y = \mathrm{f}(x)$ such that there exists one and only one value of y in the range of f for each real number x in the domain of f. The variable y is called the *dependent variable*.

An *implicit function* is a function in which both dependent- and independent variables appear on the same side.

An *explicit function* is one where the dependent variable is on the left hand side-, and the independent variable on the right hand side of the equation.

Polynomial

$$a_n x^n + \ldots + a_1 x + a_0$$

Polynomial equation

$$a_n x^n + \ldots + a_1 x + a_0 = 0$$

Quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of limits

If $\mathrm{f}(x)$ is a function which draws closer to a unique finite real number l for all values of x as the latter draws closer to a, but $x \neq a$, then l is called the *limit* of $\mathrm{f}(x)$ as x approaches a. In notation this is,

$$\lim_{n \to a} \mathrm{f}(x) = l$$

Definition of limits

For a function $\mathrm{f}(x)$, $\lim_{n\to a}\mathrm{f}(x) = l$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\left|\mathrm{f}(x) - l\right| < \epsilon$ whenever $0 < \left|x - a\right| < \delta$.

Rules of limits

$$\lim_{x \to a} k = k$$

$$\lim_{x \to a} x^n = a^n$$

$$\lim_{x \to a} k\mathrm{f}(x) = k \lim_{x \to a} \mathrm{f}(x)$$

$$\lim_{x \to a} [\mathrm{f}(x) \pm \mathrm{g}(x)] = \lim_{x \to a} \mathrm{f}(x) \pm \lim_{x \to a} \mathrm{g}(x)$$

$$\lim_{x\to a}[\mathrm{f}(x)\cdot \mathrm{g}(x)] = \lim_{x\to a}\mathrm{f}(x)\cdot\lim_{x\to a}\mathrm{g}(x)$$

$$\lim_{x\to a}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\lim_{x\to a}\mathrm{f}(x)}{\lim_{x\to a}\mathrm{g}(x)}, \text{ provided that } \lim_{x\to a}\mathrm{g}(x)\neq 0$$

$$\lim_{x\to a}[\mathrm{f}(x)]^n = \left[\lim_{x\to a}\mathrm{f}(x)\right]^n, \text{ provided that } n>0$$

Definition of a derivative

Let $y = \mathrm{f}(x)$. Then, the derivative of y with respect to x is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

The various notations for the derivative include

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x},\ \frac{\mathrm{d}f}{\mathrm{d}x},\ f'(x),\ y',\ \mathrm{D}y \text{ and } \mathrm{D}(\mathrm{f}(x))$$

General rules of differentiation

Let u, v and w are functions of x, and c is a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cx) = c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cx^n) = ncx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \pm v \pm \ldots) = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x} \pm \ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cu) = c\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uvw) = uv\frac{\mathrm{d}w}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) - u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)}{v^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u^n) = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}u}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}u}}{\frac{\mathrm{d}x}{\mathrm{d}u}}$$

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$$

Laws of exponents

Let p and q be real numbers, a and b positive numbers, and m and n positive integers. Then,

$$a^p \cdot a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$(a^p)^q = a^{pq}$$

$$a^0 = 1, \text{ provided that } a \neq 0$$

$$a^{-p} = \frac{1}{a^p}$$
$$(ab)^p = a^p b^p$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Laws of logarithms

$$\log_a mn = \log_a m + \log_a n$$
$$\log_a \frac{m}{n} = \log_a m - \log_a n$$
$$\log_a m^p = p \log_a m$$
$$\log_a n = \frac{\log_b n}{\log_b a}$$

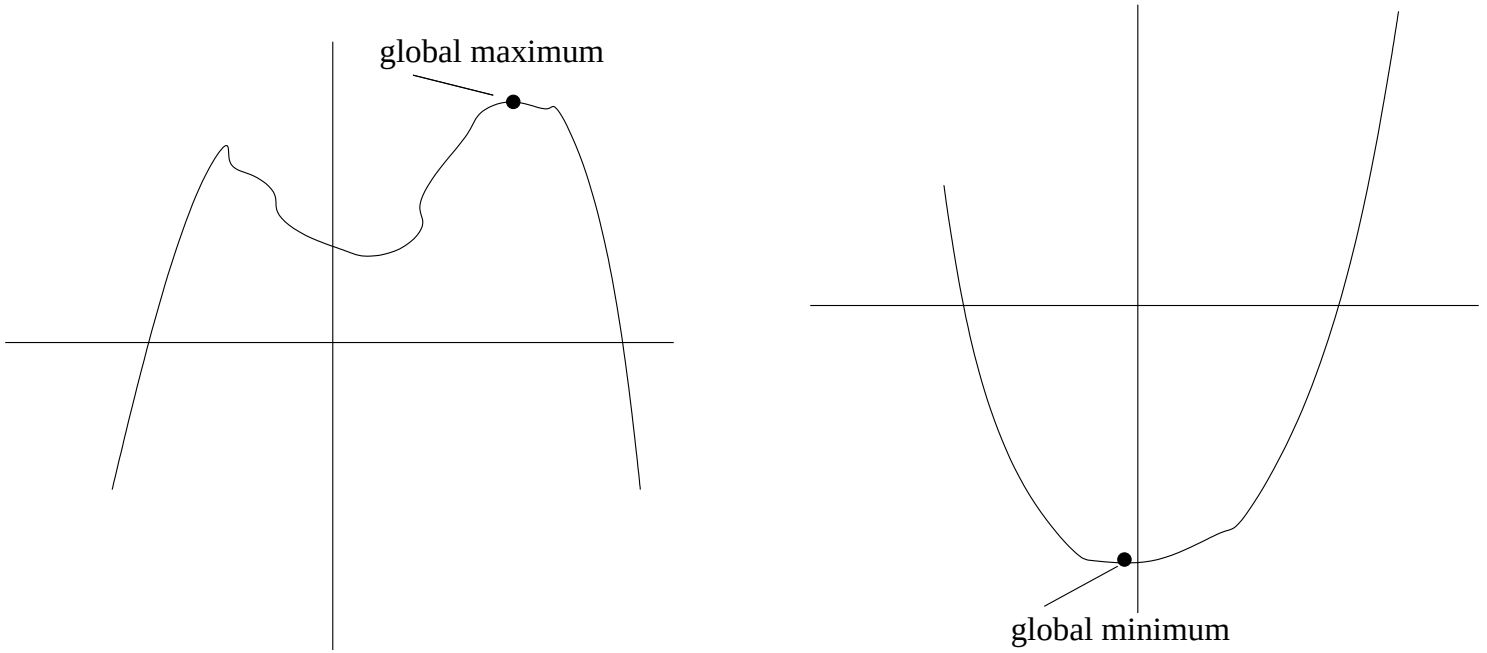

Figure 1. *(a) global maximum-, and (b) global minimum points of a function.*

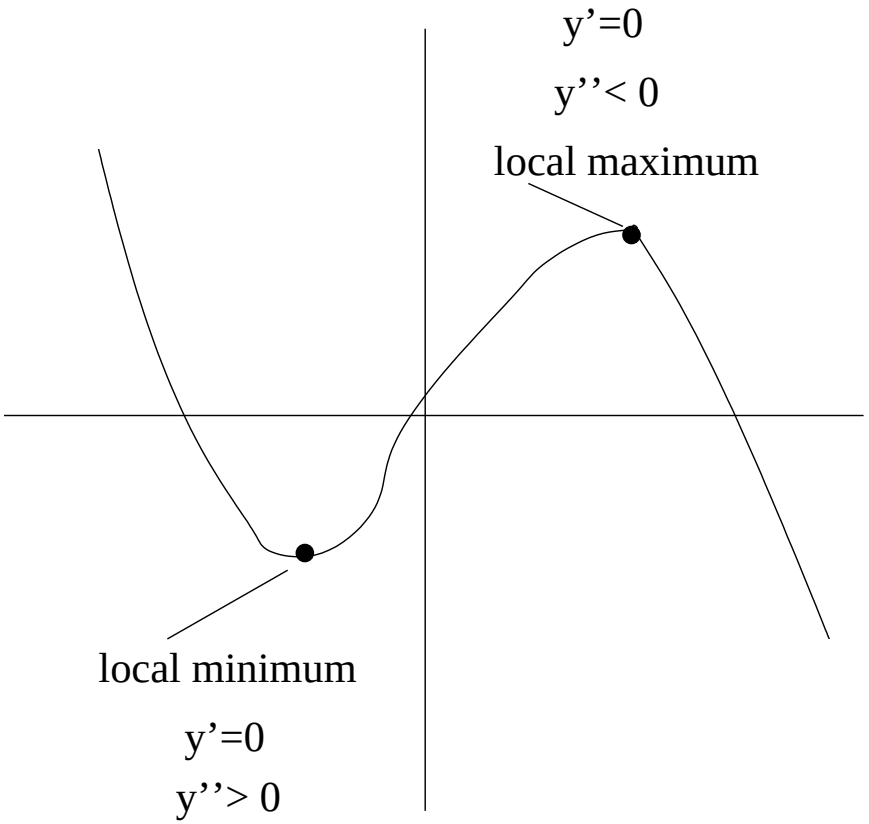

Figure 2. *Local minimum- and local maximum points of a function.*

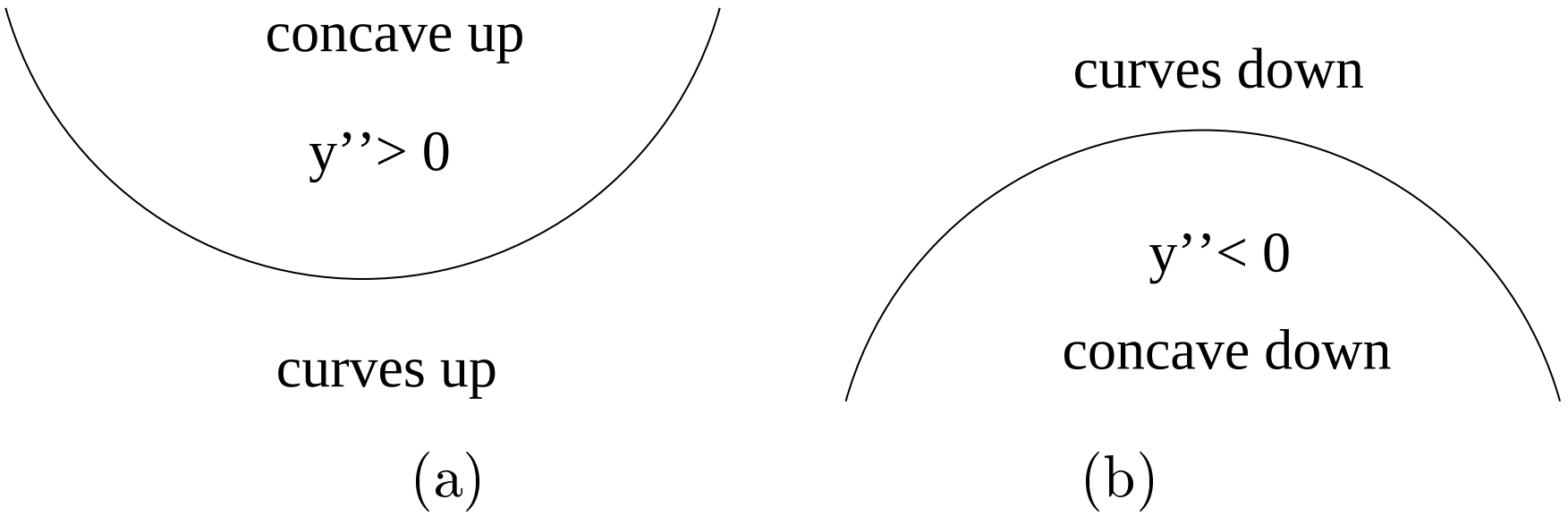

Figure 3. *The two curvature types, namely (a) concave up ($y'' > 0$) and, (b) concave down ($y'' < 0$).*

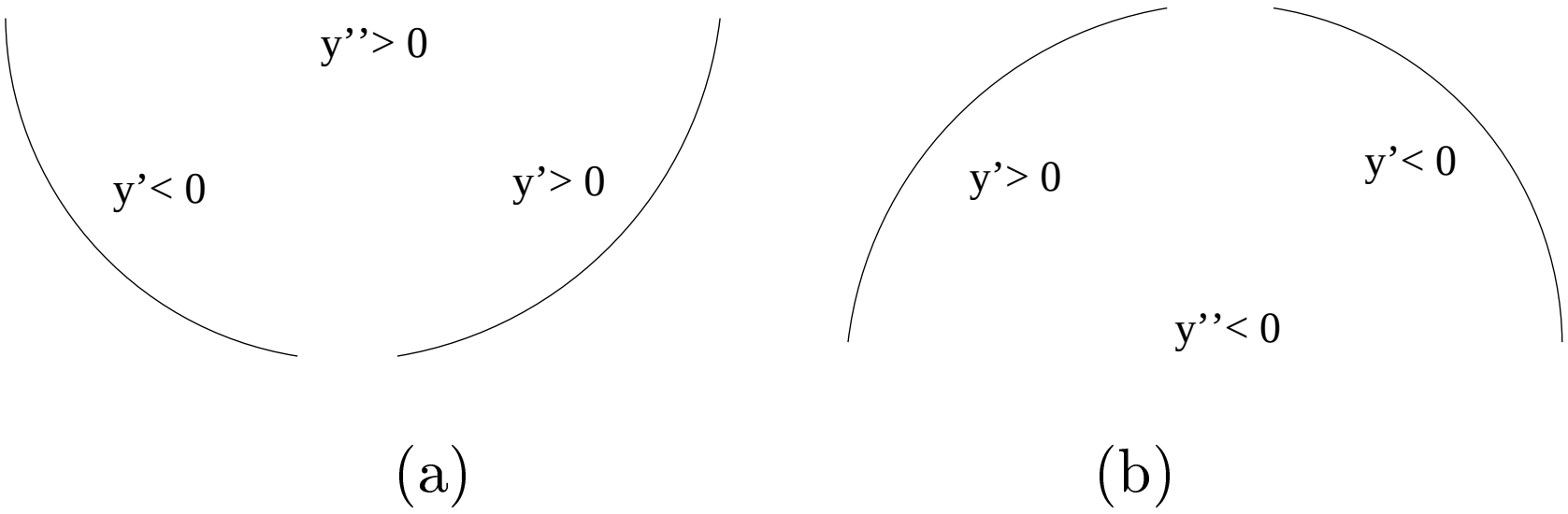

Figure 4. *The four possible curvatures in two dimensions, considering both y' and y'', (a) $y'' > 0$ and, (b) $y'' < 0$.*

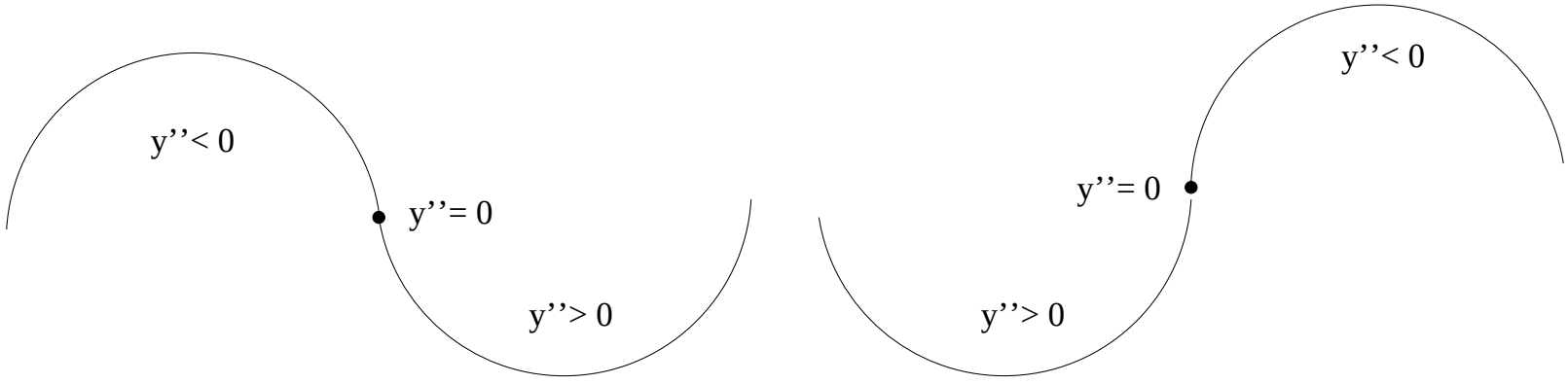

Figure 5. *Inflection points, where $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*

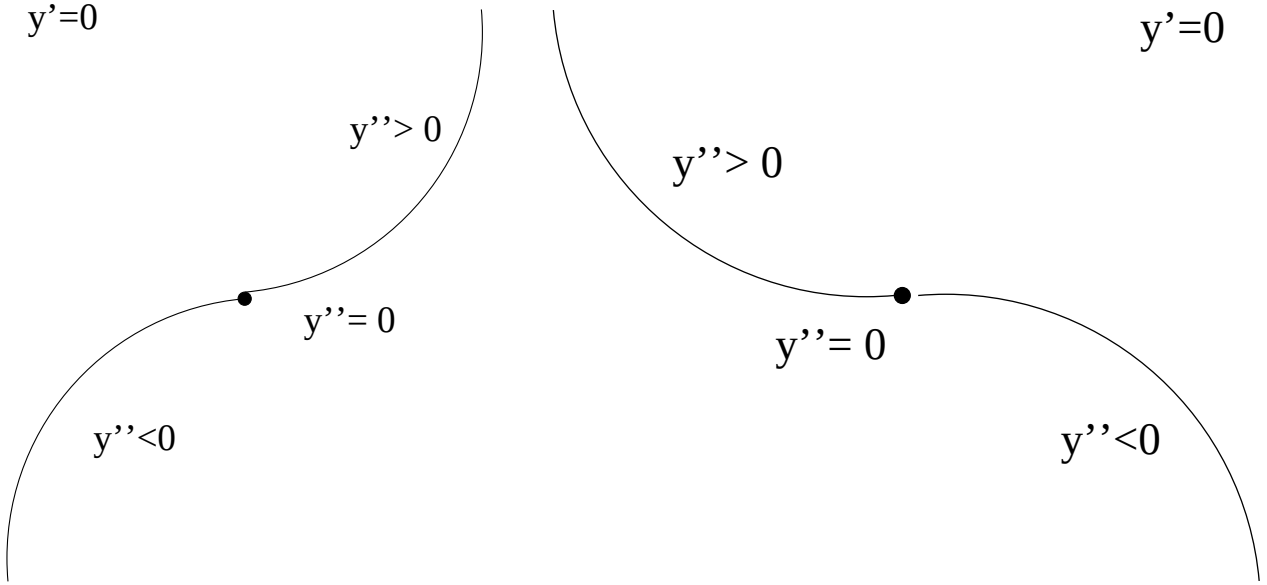

Figure 6. *Stationary inflection points, where both* $y' = 0$ *and* $y'' = 0$*, (a) with* y'' *increasing and, (b) with* y'' *decreasing.*

Figure 7 shows a plot of the case where $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

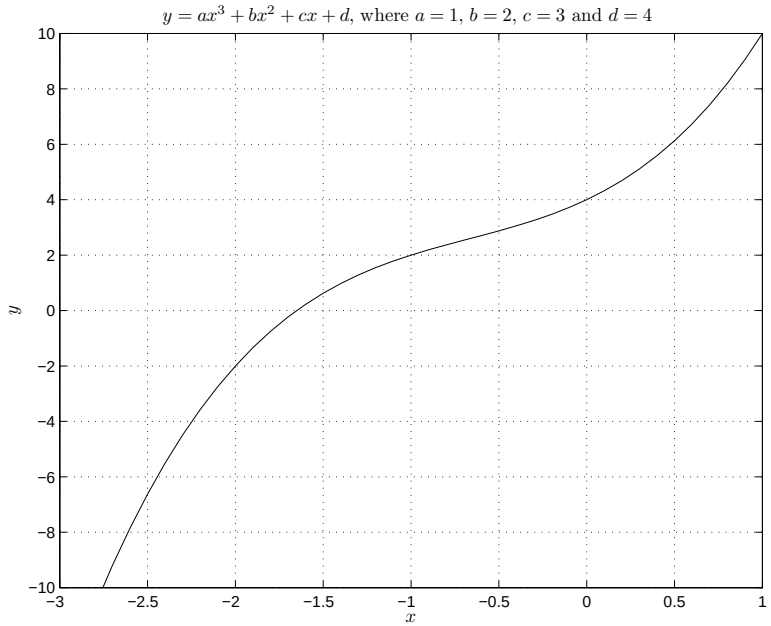

Figure 7. *The cubic function* $y = x^3 + 2x^2 + 3x + 4$.

Figure 8 shows the case where $a = 1$, $b = 5$, $c = 4$ and $d = 3$

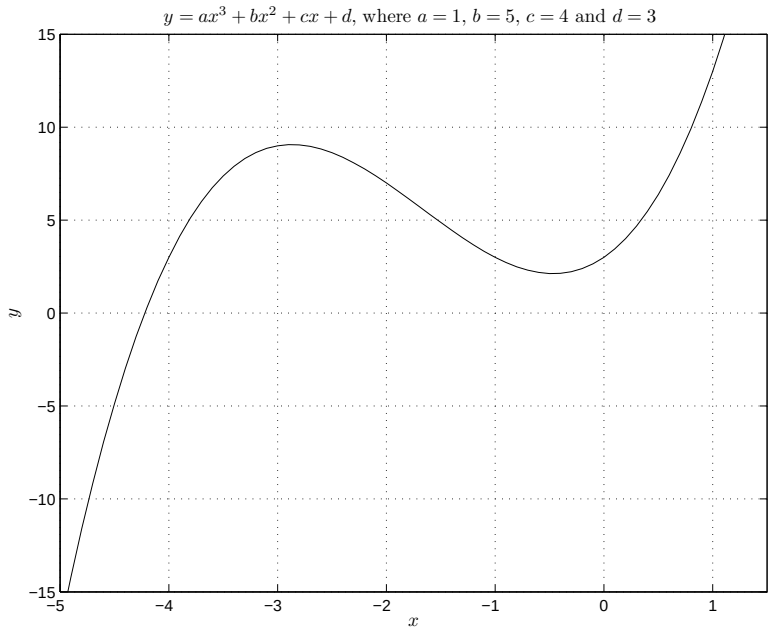

Figure 8. *The cubic function* $y = x^3 + 5x^2 + 4x + 3$.

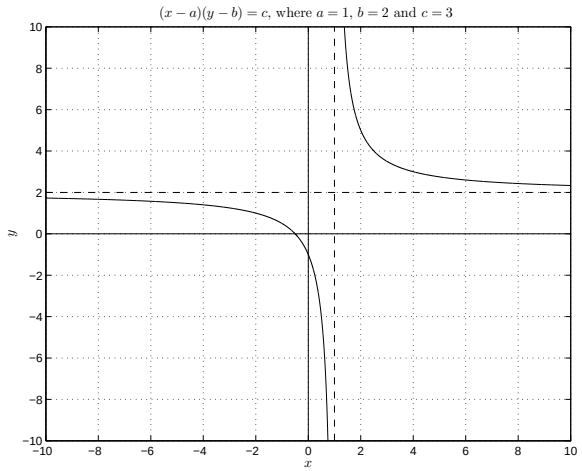

Figure 9. *The hyperbolic function* $(x-1)(y-2)=3$.

The *general demand function* is of the form

$$q_d = \mathrm{f}(p, y, p_s, p_c, t_a, a, \ldots)$$

where q_d is the quantity demand of good x, p the price of x, y the income of the consumer, p_s the price of substitute goods, p_c the price of complementary goods, t_a the taste or fashion of the consumer, and a the advertisement level.

In its simplest case where all other factors are constant, the *demand equation* takes the form

$$p = c_1 - c_2 q_d$$

where p is the price-, while q_d the quantity demanded of the good x, and c_1 and c_2 are positive constants.

The *general supply function* is of the form

$$q_s = \mathrm{f}(p, c, p_0, t_e, n, o, \ldots)$$

where q_s is the quantity supplied of good x, p the price of x, c the cost of production, p_0 the price of other goods, t_e the available technology, n the number of producers in the market, and o other factors, for example tax and subsidies.

The simplified relation for the *supply* is

$$p = c_1 + c_2 q_s$$

where q_s is the quantity of x supplied, and c_1 and c_2 are positive constants.

Definition 1. The *total cost* c_t comprises a fixed cost and a variable cost, that is

$$c_t = c_f + c_v$$

The *total revenue* r_t is price times output,

$$r_t = pq$$

Definition 2. The *marginal cost* c_m is the change in total cost caused by the production of an additional unit.

The *marginal revenue* r_m is the change in total revenue coming from the sale of an extra good. In other words,

$$c_m = \frac{\mathrm{d}c_t}{\mathrm{d}q} \text{ and}$$

$$r_m = \frac{\mathrm{d}r_t}{\mathrm{d}q}$$

where q is the output.

Definition 3. The *average cost* c_a is the total cost per unit output,

$$c_a = \frac{c_t}{q}$$

The *average revenue* r_a is the total revenue per unit output,

$$r_a = \frac{r_t}{q}$$

From Definition 1, $r_t = pq$, and from Definition 3, $r_t = r_a q$, therefore

$$p - r_a$$

From Definition 3, $c_a = \frac{c_t}{q}$, and from Definition 1, $c_a = c_f + c_v$. Therefore

$$c_a = c_{af} + c_{av}$$

where the average fixed cost $c_{af} = \frac{c_f}{q}$ and the average variable cost $c_{av} = \frac{c_v}{q}$.

Procedure 1. *Total-, average-, and marginal graphs*

Given: $c_t(x)$

$c_a \leftarrow \frac{c_t}{x}$

$c_m \leftarrow c_t'$

for each $f \in \{c_t, c_a, c_m\}$ **do**
  find the critical values $\mathbf{x}_c$ for $f' = 0$
  find f''
  $n \leftarrow |\mathbf{x}_c|$
  for $i = 1$ to n **do**
    if $f''(x_i^c) > 0$ **then**
      $f(x_i^c)$ is convex and is the relative minimum of f
    elseif $f''(x_i^c) < 0$ **then**
      $f(x_i^c)$ is concave and is the relative maximum of f
    endif
  endfor
  find inflection points $\mathbf{x}_f$ from $f'' = 0$
endfor
plot $c_t(x)$, then $c_a(x)$ and $c_m(x)$

The relationship between input and output is called a *production function*,

$$q = \mathrm{f}(l, k, r, t_e, s, e, \ldots)$$

where l is labour, k phical capital such as buildings and machines, r raw materials, t_e technology, s land, and e enterprise.

Assuming a short period of time, then l becomes the only independent variable and the other remaining factors are parameters, that is fixed, and therefore $q = \mathrm{f}(l)$.

Then the *marginal product of labour* is

$$p_{lm} = \frac{\mathrm{d}q}{\mathrm{d}l}$$

and the *average product of labour* is

$$p_{la} = \frac{q}{l}$$

The *marginal propensity to consume* is $p_{cm} = \frac{\mathrm{d}c}{\mathrm{d}y}$.

The *marginal propensity to save* is $p_{sm} = \frac{\mathrm{d}s}{\mathrm{d}y}$.

The *average propensity to consume* is $p_{ca} = \frac{c}{y}$.

The *average propensity to save* is $p_{sa} = \frac{s}{y}$.

Here y is the income, c the consumption, and s the saving.

The *profit* is

$$\pi = r_t - c_t$$

At the *break-even point*

$$\pi = 0$$

that is

$$r_t = c_t$$